INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 25, 861–878 (1997)

ASYMPTOTIC–NEWTON METHOD FOR SOLVING INCOMPRESSIBLE FLOWS

SOFIANE HADJI[1] AND GOURI. DHATT[1,2]*

[1] *Universsié de Compiègne (UTC), Génie Mécanique, F-60200 Compiègne, France*
[2] *INSA de Rouen, Génie Mécanique, Rouen, France*

SUMMARY

In this paper we present a comparative study of three non-linear schemes for solving finite element systems of Navier–Stokes incompressible flows. The first scheme is the classical Newton–Raphson linearization, the second one is the modified Newton–Raphson linearization and the last one is a new scheme called the asymptotic–Newton method. The relative efficiency of these approaches is evaluated over a large number of examples. © 1997 John Wiley & Sons, Ltd.

Int. J. Numer. Meth. Fluids, **25**: 861–878 (1997)

No. of Figures: 14. No. of Tables: 4. No. of References: 8.

KEY WORDS: Navier–Stokes; non-linear methods; asymptotic–Newton method

1. INTRODUCTION

This study deals with the evaluation of different numerical methods for solving highly non-linear incompressible flows. A Galerkin-type finite element discretization with the velocity field richer than the pressure field is employed to obtain the discretized non-linear relations. The solution strategy involves a proper choice of linearization technique and method of resolution for the corresponding linear system.

One of the first methods employed with finite volume discretization is based on the SIMPLE[1] (semi-implicit method for pressure-linked equations) technique, which involves a segregated strategy along with a Jacobi-type linearization. With present-day computing power it seems that the use of a Newton-type linearization without segregation is highly efficient.

A major drawback of the Newton–Raphson method is the required factorization of the tangent matrix for each iteration. A variant of this method is the modified Newton method, where the tangent matrix is factorized only once for a number of steps. The occasional tangent matrix update strategy may lead to poor convergence for flows with dominant convective terms. We may improve the convergence behaviour by constructing the solution space through the asymptotic representation of the residue in the form of a series expansion using a single parameterization. Such an approach has been successfully employed for non-linear flexible structures under the name of the asymptotic–numerical method.[2,3]

* Correspondence to: G. Dhatt, INSA de Rouen, Génie Mécanique, Rouen, France.

CCC 0271–2091/97/080861–18 $17.50

Received May 1996

The purpose of the present study is to investigate the efficiency of Newton-type occasional tangent matrix update, coupled with the construction of the solution space using an asymptotic representation for non-linear incompressible flows. The relative efficiency of different Newton-type methods will be assessed for a number of two-dimensional flows with different Reynolds numbers.

2. MATHEMATICAL MODEL[7,8]

Incompressible flow is governed by the following momentum and mass conservation relations:

momentum conservation (*ρ constant*)

$$\begin{aligned} &\rho\frac{\partial}{\partial x}(u^2)+\rho\frac{\partial}{\partial y}(uv)+\frac{\partial p}{\partial x}-\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\rho f_x=0,\\ &\rho\frac{\partial}{\partial x}(uv)+\rho\frac{\partial}{\partial y}(v^2)+\frac{\partial p}{\partial y}-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\rho f_y=0, \end{aligned} \tag{1a}$$

mass conservation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1b}$$

boundary conditions

$$\begin{aligned} \text{solid wall:}\quad & \vec{u}=0,\\ \text{inflow boundary:}\quad & \vec{u}\cdot\vec{n}=u_n,\\ \text{stress-free boundary:}\quad & \frac{\partial u_n}{\partial n}=0,\quad \frac{\partial u_t}{\partial n}=0,\quad p=\vec{p}, \end{aligned}$$

where $\vec{u}=\langle u,v\rangle$ are the Cartesian velocity components, u_t and u_n are the tangential and normal components at the boundary with normal $\vec{n}, p$ is the pressure, ρ is the density, μ is the viscosity, $\vartheta=\mu/\rho$ is the kinematic viscosity and $\rho\vec{f}$ is the body gravity force. The above relations may be written in the following form by grouping linear and non-linear terms:

$$Q(U,U)+L(U)=F_{\text{ext}}, \tag{2}$$

with

$$L(U)=\begin{Bmatrix} \dfrac{\partial p}{\partial x}-\mu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)\\ \dfrac{\partial p}{\partial y}-\mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)\\ -\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right) \end{Bmatrix},$$

$$Q(U,U)=\begin{Bmatrix} \rho\dfrac{\partial}{\partial x}(uu)+\rho\dfrac{\partial}{\partial y}(uv)\\ \rho\dfrac{\partial}{\partial x}(uv)+\rho\dfrac{\partial}{\partial y}(vv)\\ 0 \end{Bmatrix},\qquad F_{\text{ext}}=\begin{Bmatrix}\rho f_x\\ \rho f_y\\ 0\end{Bmatrix}.$$

The weak form associated with equation (2) is

$$W = W_L + W_Q + W_{\text{ext}} = 0, \quad \forall \delta u, \delta v, \delta p, \tag{3}$$

with

$$\begin{aligned} W_L &= \int_A (\delta \vec{u} \cdot \nabla p + \nabla \delta u \cdot \mu \nabla u + \nabla \delta v \cdot \mu \nabla v - \delta p \operatorname{div} \cdot \vec{u}) \, \mathrm{d}A, \\ W_Q &= \int_A \left[\delta u \cdot \left(\frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial x}(uv) \right) + \delta v \cdot \left(\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) \right) \right] \mathrm{d}A, \\ W_{\text{ext}} &= -\int_A (\delta u \cdot \rho f_x + \delta y \cdot \rho f_y) \, \mathrm{d}A - \int_{S_2} (\delta u_t f_t + \delta u_n f_n) \, \mathrm{d}s, \end{aligned}$$

where δu, δv and δp are Galerkin-type test functions and S_2 is the Neumann stress boundary. The discretized model associated with equation (3) is obtained by using the finite element approximation satisfying the required consistency and stability conditions:

$$W_h = W_{hL} + W_{hQ} + W_{h\text{ext}} = 0, \quad \forall \delta u_n, \delta v_n, \delta p_n,$$

or (4)

$$\{R_L(u_n)\} + \{R_Q(u_n, u_n)\} - \{F_n\} = 0,$$

with

$$W_{hL} = \langle \delta u_n \rangle \{R_L(u_n)\}, \qquad W_{hQ} = \langle \delta u_n \rangle \{R_Q(u_n, u_n)\}, \qquad W_{h\text{ext}} = -\langle \delta u_n \rangle \{F_n\},$$

where $\{u_n\}$ are the nodal variables, $\{\delta u_n\}$ are the nodal test function components and $\{F_n\}$ is the load vector including Neumann boundary conditions.

We employ a P2–P1 element for the finite element approximation: a triangle is composed of four subtriangles, the velocity field is linear on each subtriangle and the pressure field is linear on the base triangle (Figure 1). Over each subtriangle,

$$\langle u, v \rangle = \sum N_i \langle u, v \rangle_i, \qquad \langle u^2, uv, v^2 \rangle = \sum N_i \langle u^2, uv, v^2 \rangle_i, \tag{5}$$

and similarly for δu and δv. Over each base triangle,

$$p = \sum N_i p_i.$$

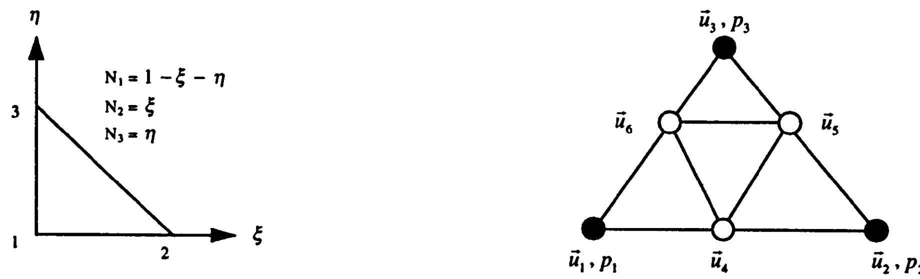


Figure 1. P2–P1 element: left, reference triangle for subtriangle or base triangle; right, six-node base triangle composed of four subtriangles

3. NEWTON-TYPE LINEARIZATION

In order to solve the set of non-linear algebraic equations (4), a certain type of linearization is necessary. In effect, one transforms the non-linear problem into a series of linear problems such that the subspace of linear solutions converges to the required non-linear solution. The success of such an iterative strategy depends on the construction of the linearized problem. It is well known that if the initial estimate is close to the exact solution, the Newton-type linearization has excellent convergence properties.

The main features of a Newton-type solution method are as follows.

1. The solution $\{U_0\}$ is known for a load level $\{F_0\}$ which may correspond to a given Reynolds number
$$\{R_0\} = \{R_L(U_0)\} + \{R_Q(U_0, U_0)\} - \{F_0\} = \{0\}. \tag{6a}$$
2. Choose the load level or Reynolds number
$$\{F\} = \{F_0\} + a\{F_1\}. \tag{6b}$$
One may fix the amplitude a or calculate it using the norm of the first solution vector.
3. Obtain the solution by solving a set of linear problems. The solution space is spanned by $\{U^{(1)}\}, \{U^{(2)}\}, \ldots, \{U^{(m)}\}$ which are obtained by solving linear systems. We then seek the solution $\{U\}$ such that
$$\{U\} = \{U_0\} + a_1\{U^{(1)}\} + a_2\{U^{(2)}\} + \cdots + a_m\{U^{(m)}\}, \tag{7a}$$
$$\{R(U)\} = \left\{R\left(U_0 + \sum_{i=1}^{m} a_i U^{(i)}\right)\right\} = 0. \tag{7b}$$
The vectors $\{U^{(i)}\}$ are solutions of linear problems
$$[K_T]\{U^{(i)}\} + \{\bar{R}^{(i)}\} = 0, \tag{8a}$$
leading to
$$\{R(U)\} = \sum ([K_T]\{U^{(i)}\} + \{\bar{R}^{(i)}\}) = \{0\}. \tag{8b}$$

The matrix $[K_T]$ is obtained by using Newton-type linearization for a known $\{U\}$. The vector $\{\bar{R}\}$ may correspond to the expression of $\{R\}$ or may correspond to an asymptotic representation. The coefficients a_i represent relaxation parameters or asymptotic expansion. Relation (7b) must be satisfied in order to obtain the desired solution.

The choice of $[K_T]$ and $\{\bar{R}\}$ leads to a family of methods which are presented in the following subsections.

3.1. Newton–Raphson method

The tangent matrix $[K_T]$ is calculated for a given estimate of the solution $\{U\}$. This is obtained by discretizing the expression ΔW which is linear in $\{\Delta U\}$:

$$W(U + \Delta U) = W(U) + \Delta W + \ldots, \tag{9a}$$

$$\Delta W_h = \langle \delta u_n \rangle [K_T]\{\Delta u\}. \tag{9b}$$

If we choose $a_i = 1$, then

$$\{U\} = \{U_0\} + \{U^{(1)}\} + \{U^{(2)}\} + \ldots, \tag{10a}$$

$$\left[K_T\left(U_0 + \sum_{j=1}^{i-1} U^{(j)}\right)\right]\{U^i\} = -\{\bar{R}\}, \tag{10b}$$

with

$$\{\bar{R}\} = \left\{R\left(U_0 + \sum_{j=1}^{i-1} U^j\right)\right\}.$$

The vectors $\{U^{(1)}\}$ and $\{U^{(2)}\}$ represents incremental vectors $\{\Delta U\}$ for each iteration. The increment a of equation (6b) may be chosen as follows.

1. The user defines the value of a.
2. The user defines the value of the norm s_0 such that

$$\|a \cdot U^{(1)}\| = s_0. \tag{11}$$

3. Adjust the value of a at each iteration level for a given norm s_0 based on the notion of arc length[4] (Figure 2).

   (a) OP_1

$$\|a \cdot U^{(1)}\| = s_0. \tag{12a}$$

   (b) OP_2

$$\begin{aligned} \text{orthogonalization:} \quad & (\Delta a U^{(1)} + U^{(2)}) \cdot U^{(1)} = 0,\ a = a + \Delta a,\ O\bar{P}_2 = OP_1 + (\Delta a U^{(1)} + U^{(2)}); \\ \text{adjustment:} \quad & OP_2 = (O\bar{P}_2/|O\bar{P}_2|)s_0. \end{aligned} \tag{12b}$$

   (c) OP_3

$$\begin{aligned} \text{orthogonalization:} \quad & (\Delta a U^{(1)} + U^{(3)}) \cdot U^{(2)} = 0,\ a = a + \Delta a,\ O\bar{P}_3 = OP_1 + (\Delta a U^{(1)} + U^{(3)}); \\ \text{adjustment:} \quad & OP_3 = (O\bar{P}_3/|O\bar{P}_3|)s_0. \end{aligned} \tag{12c}$$

The Newton–Raphson algorithm is summarized in Figure 3.

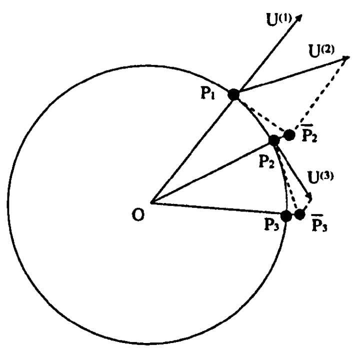


Figure 2. Adjustment of a by arc length

- Choice of convergence norm ε
- Solution $\{U_0\}$ for load level F_0
- Choice of increment norm 'a'

ITERATION: ITER = 1, 2,......., NITER

Calculate $[K_T]$
Calculate {R}
Solve: $[K_T]\{\Delta U\} = \{R\}$

For fixed 'a' or by (12):

Solution: *1 - For fixed a:*
{U} = {U} + {ΔU}
2 - Using (12), {ΔU}
a = a + Δa
$U = OP^{ITER}$

Calculate relative norm:
|| {ΔU} || / || {U} ||
|| {R} || / || a $\{F_1\}$||

Convergence:
Norm < ε

END LOOP ITER

Figure 3. Newton–Raphson algorithm

3.2. *Modified Newton–Raphson method*

This method is identical with the Newton–Raphson method except that the tangent matrix is calculated and factorized once at the beginning of iterations or maintained constant over a number of load steps. The computationally expensive operation of factoring $[K_T]$ is performed occasionally, which leads to relative efficiency of the solution method. If the convergence behaviour becomes slow or erroneous, we recalculate $[K_T]$ and perform a number of iterations to obtain the desired solution.

4. ASYMPTOTIC–NEWTON METHOD

As presented at the beginning of the last section (equations (7b)), the essential aspects of the solution method are the choice of the tangent or Jacobian matrix $[K_T]$ and the choice of the residue vector $\{\bar{R}\}$ for each iteration. In the asymptotic method we have the following.

1. The tangent matrix is calculated once at the beginning of iterations or at the beginning of a certain number of load cycles. This step is then identical with the modified Newton–Raphson method.
2. The evaluation of the residue vector is done in a special manner which differs from that of the residue vector employed for the Newton–Raphson or the modified Newton–Raphson method.

A non-linear problem is represented by a set of linear problems. The solution is written in series form with respect to a load parameter a:

$$\begin{aligned}\{F\} &= \{F_0\} + a\{F_1\},\\ \{U\} &= \{U_0\} + a\{U^{(1)}\} + a^2\{U^{(2)}\} + a^3\{U^{(3)}\} + \cdots + a^m\{U^{(m)}\}.\end{aligned} \tag{13a}$$

The expression for the residue becomes

$$\{R(U)\} = \{R_0\} + a\{R^{(1)}\} + a^2\{R^{(2)}\} + a^3\{R^{(3)}\} + \cdots + a^m\{R^{(m)}\}, \tag{13b}$$

which leads to a set of linear problems associated with each power series term:

$$\begin{aligned}
\{R_0\} &= \{0\},\\
\{R^{(1)}\} &= [K_{T0}]\{U^{(1)}\} - \{F_1\} = 0,\\
\{R^{(2)}\} &= [K_{T0}]\{U^{(2)}\} + \{\bar{R}^{(2)}\} = 0,\\
&\vdots\\
\{R^{(m)}\} &= [K_{T0}]\{U^{(m)}\} + \{\bar{R}^{(m)}\} = 0.
\end{aligned} \tag{14}$$

The typical feature of the asymptotic method lies in the evaluation of the residue vector $\{\bar{R}^{(i)}\}$. The residue vector in the Newton method corresponds to the expression of W for an estimated solution $\{U\}$. In the asymptotic method the residue vector corresponds to the expression related to $a, a^2, \dots, a^m$. Let us take a single-degree-of-freedom non-linear algebraic equation to explain the method.

4.1. Single-variable non-linear equation

A quadratic non-linear equation is defined by

$$L(U) + Q(U, U) - F = 0$$

or

$$k_0 U + k_1 U^2 - (F_0 + aF_1) = 0. \tag{15}$$

One supposes that the solution U_0 for a given F_0 is known:

$$R_0 = k_0 U_0 + k_1 U_0^2 - F_0 = 0.$$

For the modified Newton–Raphson method the series of linear problems is (U_m corresponds to $U^{(m)}$)

$$K_{T0} = k_0 + 2k_1 U_0, \tag{16a}$$

$$\begin{aligned}
R_{1L} &= K_{T0}U_1 + \underbrace{k_0U_0 + k_1U_0^2 - F_0 - aF_1}_{R_1} = K_{T0}U_1 \underbrace{-aF_1 = 0}_{R_1} \quad (\Delta U = U_1),\\
R_{2L} &= K_{T0}U_2 + \underbrace{k_0(U_0+U_1) + k_1(U_0+U_1)^2 - F_0 - aF_1}_{R_2} = K_{T0}U_2 + \underbrace{k_1U_1^2}_{R_2} = 0,\\
U &= U_0 + U_1 + U_2,\\
R_{3L} &= K_{T0}U_3 + \underbrace{kU + kU^2 - F_0 - aF_1}_{R_3} = K_{T0}U_3 + \underbrace{k_1(2U_1U_2 + U_1^2)}_{R_3} = 0,\\
U &= U_0 + \sum_{j=1,3} U_j,\\
&\vdots\\
R_{mL} &= K_{T0}U_m + \underbrace{kU + kU^2 - F_0 - aF_1}_{R_m} = K_{T0}U_m + \underbrace{k_1\left(2\sum_{1}^{m-2} U_j \sum_{j+1}^{m-1} U_i + U_{m-1}^2\right)}_{R_m} = 0,\\
U &= U_0 + \sum_{j=1,m} U_j.
\end{aligned} \tag{16b}$$

For the asymptotic–Newton method (U_m defines $U^{(m)}$) we have

$$U = U_0 + aU^{(1)} + a^2U^{(2)} + \cdots + a^mU^{(m)},\\ F = F_0 + aF_1 \tag{17a}$$

and the series of linear problems is

$$R = R_0 + aR_1 + a^2R_2 + \cdots + a^mR_m, \tag{17b}$$

with

$$\begin{aligned}
&R_0 = k_0U_0 + k_1U_0^2 - F_0 = 0,\\
&R_{1L} = K_{T0}U_1 + +\bar{R}_1 = 0, \quad \bar{R}_1 = -F_1 \quad \text{(identical with MN − R method)},\\
&R_{2L} = K_{T0}U_2 + \bar{R}_2 = 0, \quad \bar{R}_2 = k_1U_1^2 \quad \text{(identical with MN − R method)},\\
&R_{3L} = K_{T0}U_3 + \bar{R}_3 = 0, \quad \bar{R}_3 = 2k_1U_1U_2 \quad \text{(different from MN − R method)},\\
&\vdots\\
&R_{mL} = K_{T0}U_m + \bar{R}_m = 0, \quad \bar{R}_m = k_1\sum_{j=1}^{m-1} U_jU_{m-1-j}.
\end{aligned}$$

In Figures 4(a)–4(c) we represent the essential characteristics of the Newton–Raphson (N–R), modified Newton–Raphson (MN–R) and asymptotic–Newton (A–N) methods respectively.

Example (N–R method)

$$\begin{aligned}
[K_1(U_0)]\{U_1\} &= a\{F_1\},\\
[K_2(U_0, U_1)]\{U_2\} &= -\{Q(U_1, U_1)\} = -\{R_2\},\\
[K_3(U_0, U_1, U_2)]\{U_3\} &= -\{Q(U_2, U_2)\} = -R_3.
\end{aligned}$$

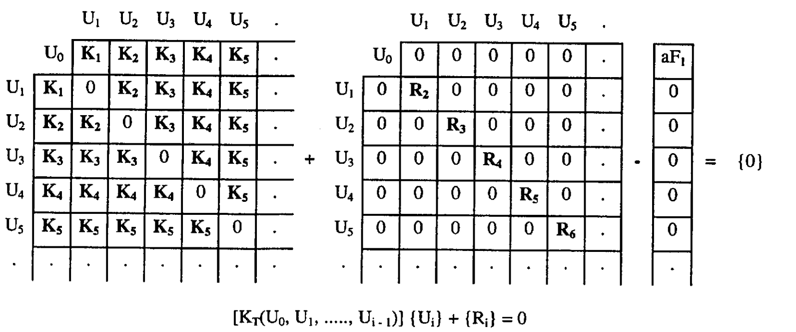
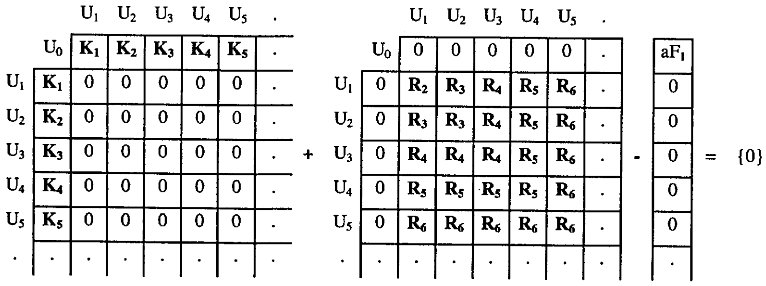
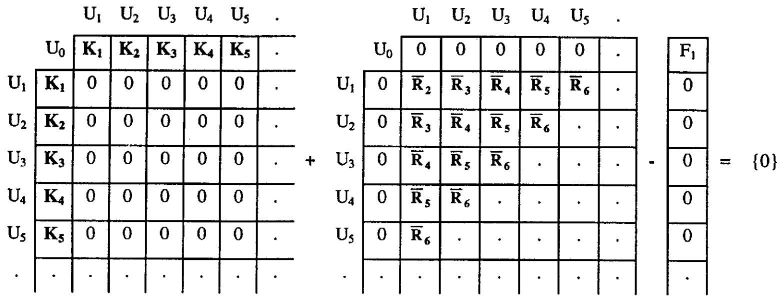


Figure 4(a). Representation of Newton–Raphson method



$[K_T(U_0)]\,\{U_i\} + \{R_i\} = 0$

Figure 4(b). Representation of modified Newton–Raphson method



$[K_T(U_0)]\,\{U_i\} + \{\bar{R}_i\} = 0$

Figure 4(c). Representation of asymptotic–Newton method

Remark. The residue vector for the Newton–Raphson method corresponds to discretization of $Q(U, U)$ with $U = \Delta U$ at the last iteration.

Example (MN–R method)

$$
\begin{aligned}
&[K_1(U_0)]\{U_1\} = \{aF_1\},\\
&[K_2(U_0)]\{U_2\} = -\{Q(U_1, U_1)\} = -\{R_2\},\\
&[K_3(U_0)]\{U_3\} = \{2Q(U_1, U_2)\} = -\{R_3\},\\
&K_1 = K_2 = K_3 = \cdots = K_m.
\end{aligned}
$$

Example (A–N method)

$$
\begin{aligned}
&[K_1(U_0)]\{U_1\} = \{F_1\},\\
&[K_2(U_0)]\{U_2\} = -\{Q(U_1, U_1)\} = -\{\bar{R}_2\},\\
&[K_3(U_0)]\{U_3\} = -\{2Q(U_1, U_2)\} = -\{\bar{R}_3\} \quad \text{(different from } R_3 \text{ of MN−R method)},\\
&K_1 = K_2 = K_3 = \cdots = K_m.
\end{aligned}
$$

4.2. Asymptotic–Newton method for Navier–Stokes flow

The numerical implementation of the asymptotic method for incompressible flow is straightforward. We have

$$\begin{aligned} u &= u_0 + au_1 + a^2u_2 + \cdots + a^m u_m, \\ v &= v_0 + av_1 + a^2v_2 + \cdots + a^m v_m, \\ p &= p_0 + ap_1 + a^2p_2 + \cdots + a^m p_m, \\ \vec{f} &= \vec{f}_0 + a\vec{f}_1. \end{aligned} \tag{18}$$

Substitution of the above expansion into equation (3) followed by finite element discretization leads to the asymptotic representation of the residue vector as

$$\{R\} = \{R_0\} + a\{R_1\} + a^2\{R_2\} + \cdots + a^m\{R_m\},$$

where $R_0, R_1, R_2, \ldots, R_m$ corresponds to discretization of equation (3).

Residue R_0

$$\begin{aligned} \{R_0\}:\quad & \sum \int_{A^e} [\delta\vec{u}\cdot\nabla p_0 + \mu_0(\nabla\delta u\cdot\nabla u_0 + \nabla\delta v\cdot\nabla v_0) - \delta p\ \mathrm{div}\cdot\vec{u}_0]\,\mathrm{d}A \\ & + \int_{A^e}\left[\delta u\left(\frac{\partial A_0}{\partial x} + \frac{\partial B_0}{\partial y}\right) + \delta v\left(\frac{\partial B_0}{\partial x} + \frac{\partial C_0}{\partial y}\right)\right]\mathrm{d}A + W_{\mathrm{ext}} = 0, \end{aligned} \tag{19}$$

where

$$A_0 = \sum N_i(A_0)_i,\qquad B_0 = \sum N_i(B_0)_i,\qquad C_0 = \sum N_i(C_0)_i,$$
$$(A_0)_i = (u_0\cdot u_0)_i,\qquad (B_0)_i = (u_0\cdot v_0)_i,\qquad (C_0)_i = (v_0\cdot v_0)_i$$

(see equation 5) and N_i are element approximation functions.

Residue R_1

$$\{R_1\} = [K_{\mathrm{T0}}]\{U_1\} - a\{F_1\} = 0. \tag{20a}$$

$[K_{\mathrm{T0}}]$ is the tangent matrix at $(\vec{u}_0, p_0)$ which corresponds to

$$\begin{aligned} [K_{\mathrm{T0}}]\{U_1\}:\quad & \sum \int_{A^e} (\delta\vec{u}\cdot\nabla p_1 + \mu_0(\nabla\delta u\cdot\nabla u_1 + \nabla\delta v\cdot\nabla v_1) - \delta p\ \mathrm{div}\cdot\vec{u}_1)\ \mathrm{d}A \\ & + \int_{A^e}\left[\delta u\left(\frac{\partial(2u_0u_1)}{\partial x} + \frac{\partial(u_0v_1 + u_1v_0)}{\partial y}\right) + \delta v\left(\frac{\partial(u_0v_1 + u_1v_0)}{\partial x} + \frac{\partial(2v_0v_1)}{\partial y}\right)\right]\ \mathrm{d}A. \end{aligned} \tag{20b}$$

$\{F_1\}$ corresponds to the load vector associated with $\vec{f}_1$, modified by the increment of boundary conditions or the increment to viscosity depending on the Reynolds increment strategy. The approximations of quadratic terms are

$$(u_0u_1) = \sum N_i(u_0)_i\cdot(u_1)_i,\qquad (v_0v_1) = \sum N_i(v_0)_i\cdot(v_1)_i,\qquad (u_0v_1) = \sum N_i(u_0)_i\cdot(v_1)_i. \tag{20c}$$

Residue R_m

$$\{R_m\} = [K_{T0}]\{U_m\} - \{\bar{R}_m\} = \{0\}. \tag{21a}$$

The residue $\{\bar{R}_m\}$ is the discretization of the following expression associated with quadratic terms:

$$\{\bar{R}_m\}: \quad \sum_{\text{elements}} \int_{A^e} \left[\delta u \left(\frac{\partial A_m}{\partial x} + \frac{\partial B_m}{\partial y} \right) + \delta v \left(\frac{\partial B_m}{\partial x} + \frac{\partial C_m}{\partial y} \right) \right] \mathrm{d}A. \tag{21b}$$

The finite approximations are $(A_m, B_m, C_m) = \sum N_i(A_m, B_m, C_m)_i$ and the terms A_m, B_m and C_m are as given by equation (17b):

$$A_m = \sum_{j=1}^{m-1} u_j \cdot u_{m-1-j}, \qquad B_m = \sum_{j=1}^{m-1} u_j \cdot v_{m-1-j}, \qquad C_m = \sum_{j=1}^{m-1} v_j \cdot v_{m-1-j}. \tag{21c}$$

Remark. Equation (21c) represents the essential expression of the asymptotic–Newton method.

4.3. Implementation of asymptotic–Newton method

In order to obtain the solution using the asymptotic method, one has to store the vectors of the solution space of size m:

$$\{U_0\}, \{U_1\}, \ldots, \{U_m\}.$$

The tangent matrix for each step is maintained constant and the residue is calculated using equation (20a) for the first vector and equation (21b) for the following vectors. The calculation is straightforward once A_m, B_m and C_m are obtained at each node using equation (20c).

Different aspects of the solution strategy employed are described below.

Incrementation of Reynolds number

The solution may be obtained by choosing the viscosity or velocity conditions corresponding to a given Reynolds number in a single step. However, for complicated flows the solution is obtained by incrementing the Reynolds number in steps. One may thus have at the boundary nodes

$$u = u_0 + au_1, \qquad v = v_0 + av_1 \tag{22a}$$

or, keeping the velocity conditions constant and varying the viscosity,

$$\mu = \mu_0 + a\mu_1, \tag{22b}$$

thus obtaining $\bar{R}_1$.

Choice of parameter a

The choice of a is controlled by limiting the size of the first vector $\{U_1\}$ for a given size s_0:

$$a\|U_1\| = s_0. \tag{23a}$$

If the solution $\{U_0\}$ does not lead to $\{R_0\} = \{0\}$, we may evaluate $\{U_1\}$ using

$$[K_{T0}]\{U_1\} = \{\bar{R}_1\} + \{R_0/a\} \tag{23b}$$

and evaluate a by

$$a\|\bar{U}_1 + U_1^{(0)}/a\| = s_0, \qquad \{U_1\} = \{\bar{U}_1\} + \{U_1^{(0)}/a\}. \tag{23c}$$

- $\{U_0\} = \{0\}$, s_0: vector size or 'a' given
- Choice of convergence norm ε

ISTEP: Re

Increment viscosity or velocity conditions

CYCLE: ICYCLE = 1,.........., NCYCLE

1 - Calculate $[K_{T0}]$

2 - Factorise [L][S]

SOLUTION SPACE: J_m = 1,.........., m

1 - Calculate (A_m, B_m, C_m) at each node (21-c)

2 - Calculate residue (21-b)

3 - Solve: $[L][S]\{U_{Jm}\} = -\{\bar{R}_m\}$

IF (J_m = 1): Calculate 'a' (23-a)

IF $\| a_m \bar{R}_m \| < \varepsilon$, GOTO update

END LOOP J_m

3 - Update $\{U_0\} = \{U_0\} + a\{U_1\} + a^2\{U_2\} + ... + a^m\{U_m\}$

4 - Calculate $\{R_0\}$

5 - IF $\| R_0 \| < \varepsilon$, GOTO Next Step

END LOOP CYCLE

END LOOP ISTEP

Figure 5. Asymptotic–Newton algorithm

Size of solution space

The choice of the number m of solution vectors may be controlled by the residue norm. However, if m becomes relatively large, one may restart the solution sequence by updating the initial vector:

$$\{U\} = \{U_0\} + a\{U_1\} + a^2\{U_2\} + \cdots + a^m\{U_m\}.$$

The asymptotic–Newton algorithm is summarized in Figure 5.

5. NUMERICAL EXAMPLES

In this section we compare the Newton–Raphson, modified Newton–Raphson and asymptotic–Newton methods for solving incompressible flows with different Reynolds numbers. The numerical experimentation is undertaken to study the following aspects:

(a) the quality of the solution obtained by the asymptotic method for different Reynolds numbers
(b) the convergence of the solution related to the number of vectors
(c) influence of cycles (Figure 5) on the convergence for different Reynolds numbers.

Three typical examples are chosen which are extensively studied by finite element researchers. Description of these examples are given in the following subsection. All examples are tested on a VAX DEC ALPHA 3000-300-X machine. The unsymmetrical tangent matrix is stored using skyline organization and factorized in the form $[K_T]=[L][S]$.[5]

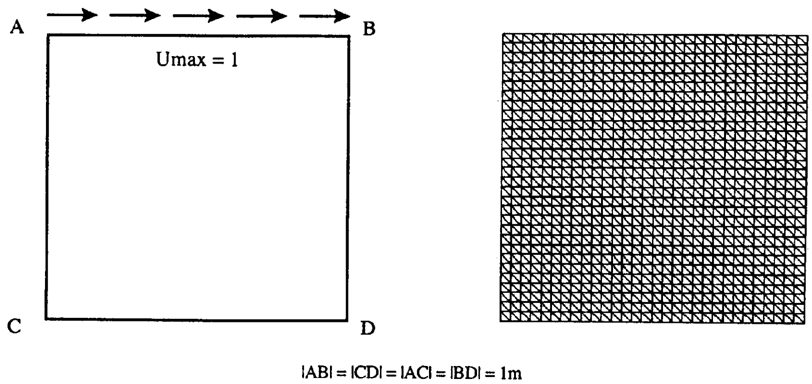


Figure 6. Geometry and mesh for square cavity

5.1. Problem descriptions

Square cavity

A square cavity with an upper lid sliding at $1\,\mathrm{m\,s^{-1}}$ is discretized using a 30×30 triangular mesh (Figure 6). The physical parameters are as follows: parameter, $\rho = 1\,\mathrm{kg\,m^{-3}}$; initial solution, $u = v = p = 0$; boundary conditions—on AB, $u = 1, v = 0$; on CD, AC, BD, $u = 0, v = 0$; at A, $p = 0$; viscosity, $\mu = 0{\cdot}1$–$0{\cdot}001$; Reynolds number, $Re = 10$–1000 in five increments; number of unknowns, 8500.

Step channel

The flow in a backward-facing channel is studied using a triangular mesh for $Re = 0$–1200 (Figure 7). The geometrical parameters are given in Reference 6. The physical parameters are as follows: parameter, $\rho = 1\,\mathrm{kg\,m^{-3}}$; initial solution, $u = v = p = 0$; boundary conditions—on AB, $u = -4y^2 + 4y$, leading to $u_{y=0{\cdot}5} = 1, v = 0$; on AD, DC, CE, FB, $u = 0, v = 0$; on EF, $v = 0, p = 0$; viscosity, $\mu = 0{\cdot}005$–$0{\cdot}000833$; Reynolds number, $Re = 0$, 200–1200 in six increments; number of unknowns, 8800.

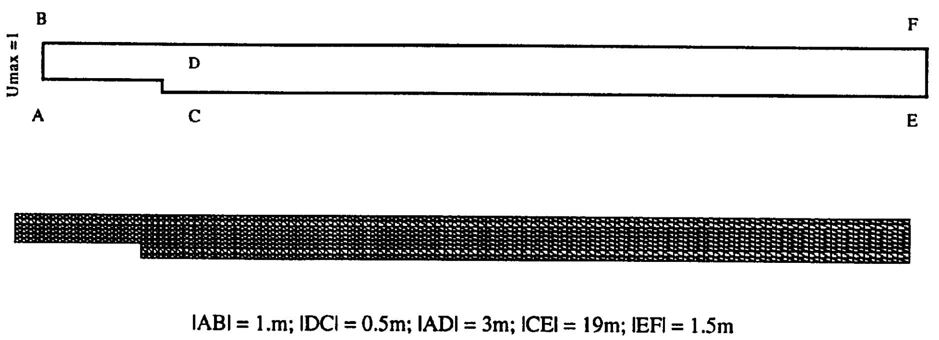


Figure 7. Geometry and mesh for step channel

Remark. The flow is of Poiseuille type near AB; we have $\partial p/\partial x = -8\mu$ in this zone.

Convoluted channel

This problem is studied for $Re = 10$–200 with 8000 unknowns (Figure 8). The geometrical parameters are given in Reference 6. The physical parameters are as follows: parameter, $\rho = 1\,\text{kg}\,\text{m}^{-3}$; initial solution, $u = v = p = 0$; boundary conditions—on AB; $u = -36y^2 + 60y - 24$ ($u_{\text{max}} = 1$), $v = 0$; on BC, CE, EI, IJ, JL, KG, GH, HF, FD, DA, $u = 0, v = 0$; on KL, $v = 0, p = 0$; viscosity, $\mu = 0{\cdot}033$–$0{\cdot}00166$; Reynolds number, $Re = 0$, 20–200 in 10 increments; number of unknowns, 8000.

5.2. Velocity and pressure profiles

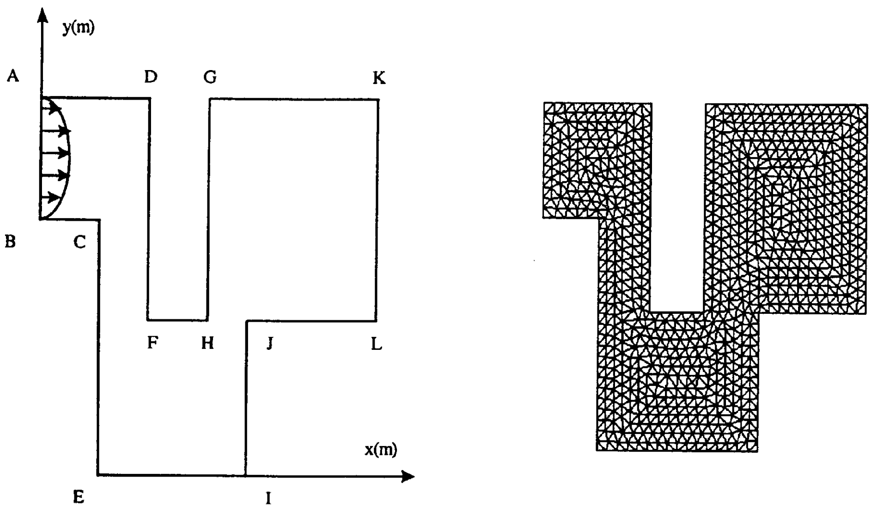
The solution obtained by the Newton–Raphson method is considered as reference. A convergence norm of $\|\Delta U\|/\|U\| < 10^{-4}$ is employed to control the precision. The step size and iteration numbers for the three problems with Newton–Raphson are given in Table I.

Typical velocity and pressure profiles for selected Reynolds numbers are presented in Figures 9–11. The results obtained compare well with those reported in Reference 7. The zones of recirculation are well captured.

5.3. Validation of asymptotic–Newton method

The converged results using the asymptotic–Newton method are found to be identical with the results obtained by the Newton–Raphson method.

We study the influence of the size of the solution space and the number of cycles on the convergence behaviour of the method.



|AB| = 0.33; |BC| = 0.17; |CE| = 0.66; |EI| = 0.5; |IJ| = 0.4; |JL| = 0.33
|LK| = 0.6; |KG| = 0.5; |GH| = |DF| = 0.5; |FH| = 0.17; |AD| = 0.33

Figure 8. Geometry and mesh for convoluted channel

Table I. Step size and iteration numbers for the three problems

Problem	Number of steps	Reynolds number increment	Number of tangent decompositions
Square cavity	5	10, 200, 400, . . . , 1000	15
Step channel	6	0, 200, 400, . . . , 1200	18
Convoluted channel	10	0, 20, 40, . . . , 200	30

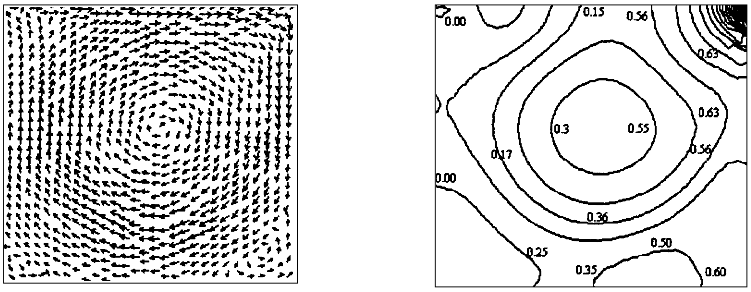


Figure 9. Velocity and pressure profiles for square cavity for $Re = 1000$

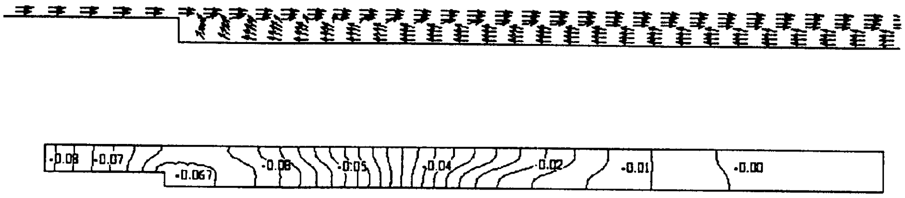


Figure 10. Velocity and pressure profiles for step channel for $Re = 1000$

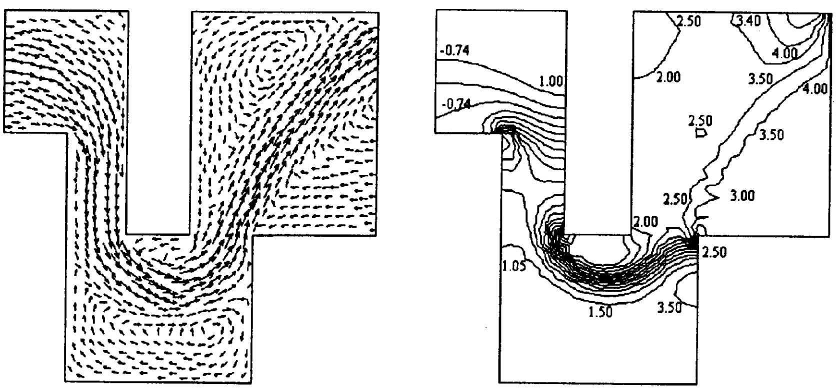


Figure 11. Velocity and pressure profiles for convoluted channel for $Re = 200$

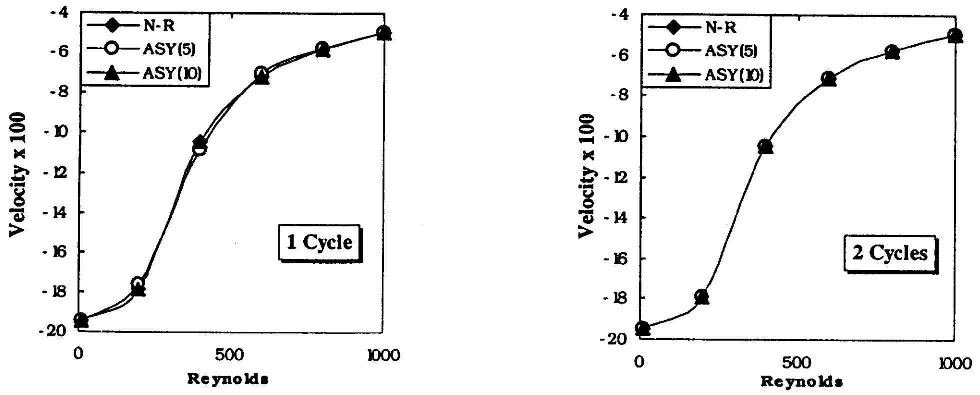


Figure 12. Velocity *u* for square cavity: 1 cycle, tangent matrix constant over each step; 2 cycles, one update of tangent matrix over each step

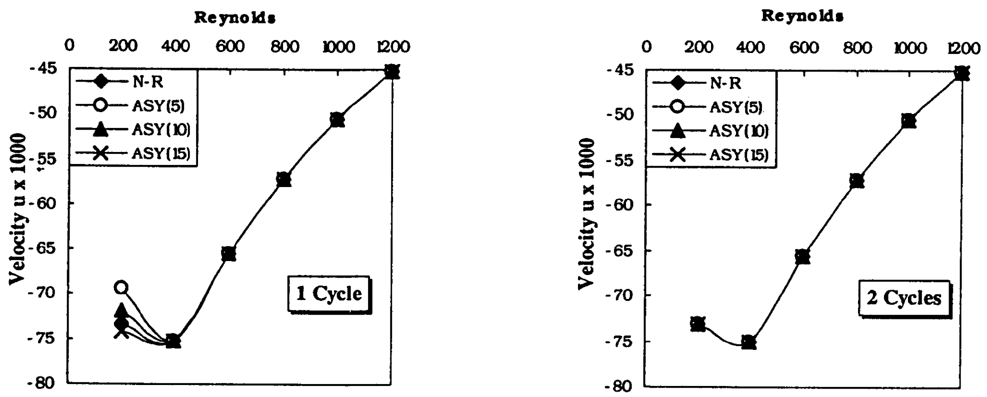


Figure 13. Velocity *u* for step channel

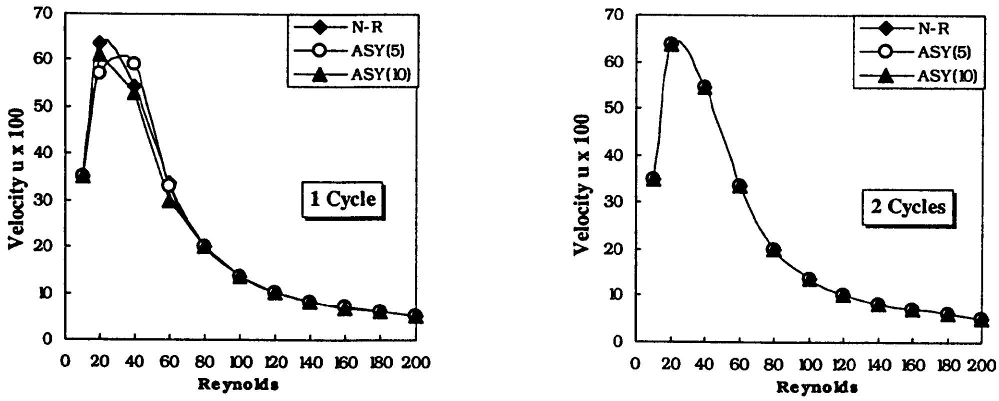


Figure 14. Velocity *u* for convoluted channel

Table II. Choice of parameters for asymptotic–Newton method

		NCYCLE = 1		*NCYCLE* = 2	
Problem (steps)	*Re* increment	Number of factorizations	Solution space *m*	Number of factorizations	Solution space *m*
Square cavity (5)	0, 200, …, 1000	5	5, 10	10	5, 10
Step channel (6)	0, 200, …, 1200	6	5, 10, 15	12	5, 10, 15
Convoluted channel (10)	0, 20, …, 200	10	5, 10	20	5, 10

Table III. CPU time in seconds for factorization and residue calculation

Problem	Factorization	One residue
Square cavity	235	9
Step channel	24	5
Convoluted channel	166	9

The solution is obtained in an incremental manner by increasing the Reynolds number in steps. In Figures 12–14 we present the influence of the size of the asymptotic space on the convergence. The velocity values for the three problems are chosen at the points

$$\begin{aligned} \text{square cavity:} &\quad x = 0{\cdot}50, \quad y = 0{\cdot}500; \\ \text{step channel:} &\quad x = 1{\cdot}39, \quad y = 0{\cdot}166, \\ \text{convoluted channel:} &\quad x = 0{\cdot}40, \quad y = 0{\cdot}143. \end{aligned}$$

In Table II we summarize the convergence aspects of the solution with the parameter $a = 1$.

Using a single cycle for each step size, one obtains acceptable results even with a five-vector solution space. For a certain value of the Reynolds number a solution space with 10 vectors improved the quality of the solution.

In another series of numerical validation experiments we employed two cycles with a solution space composed of five vectors. Excellent results are obtained which are identical with those obtained by the Newton–Raphson method.

It has been found that a solution space of five vectors with gradual incrementation of the Reynolds number is the best strategy.

The computational cost of the asymptotic–Newton method is much less than that of the Newton–Raphson method. The cost of the Newton–Raphson method for each iteration involves factorization of $[K_T]$ and calculation of the residue vector (Table III).

Table IV. CPU time in seconds for $Re = 200$

		A–N method			
		NCYCLE = 1		*NCYCLE* = 2	
Problem	N–R method	$m = 5$	$m = 10$	$m = 5$	$m = 10$
Square cavity	994	275	322	521	610
Step channel	141	54	78	82	110
Convoluted channel	710	209	278	390	440

The computation of the residue vector for different sizes of the solution space is approximately the same as with a single vector (see equation (21)).

For all practical purposes the computational cost is related to the factorization of the tangent matrix. An economy by a factor of three is obtained using the asymptotic–Newton method with $m = 5$ as compared with the Newton–Raphson method (Table IV).

6. CONCLUSIONS

In this study we have presented a new method called the asymptotic–Newton method for solving highly non-linear incompressible flows. A presentation of the method for calculating the residue vector for a given space size is given for a simple case and for a general finite element formulation of the Navier–Stokes equations.

One may consider the asymptotic–Newton method as an improvement of the modified Newton–Raphson method. The difference resides essentially in the calculation of successive residue vectors. The numerical experimentation presented in this work and that given in Reference 7 have demonstrated that the asymptotic–Newton method is a powerful tool for any computer code based on a Newton-type linearization.

Work is under way to develop a technique for obtaining the optimal choice of load parameter a and size of increment vector $\|F_1\|$. Application of this method to free surface hydraulic flows is also under way. It is equally useful to explore the strategy of tangent matrix updating such that the updates are kept to a minimum.

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